

# The Shapiro Conjecture: Prompt or Delayed Collapse in the head-on collision of neutron stars?

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We study the question of prompt vs. delayed collapse in the head-on collision of two neutron stars. We show that the prompt formation of a black hole is possible, contrary to a conjecture of Shapiro which claims that collapse is delayed until after neutrino cooling. We discuss the insight provided by Shapiro's conjecture and its limitation. An understanding of the limitation of the conjecture is provided in terms of the many time scales involved in the problem. General relativistic simulations in the Einstein theory with the full set of Einstein equations coupled to the general relativistic hydrodynamic equations are carried out in our study.

**Introduction.** The study of the coalescence of neutron stars (NSs) is important for gravitational wave astronomy and high energy astronomy. However, at present we lack even a qualitative understanding of the process. One issue is the prompt vs. delayed collapse problem. While we expect that two  $1.4 M_{\odot}$  NSs when merged will eventually collapse to form a black hole, the collapse could be delayed by fragmentation/mass shredding, angular momentum hang-up, and/or shock heating. The time scale of the collapse has important implications on the gravitational wave signals to be detected by LIGO [1]. We focus on the issue of prompt vs. delayed collapse in this paper.

The difficulty of getting an answer to this aspect of NS coalescence is very much the same as the full coalescence problem. Namely, we need to solve the full Einstein equations coupled to the general relativistic hydrodynamic (GR-hydro) equations.

Recently, Shapiro [2] put up an argument suggesting that one may be able to answer this question without numerical simulations, at least for the case of head-on collisions. The ‘‘Shapiro conjecture’’ goes as follows: Given the conditions: (I) that the two NSs are colliding head-on after falling in from infinity, and (II) the NSs are described by a polytropic equation of state (EOS)  $P = K\rho^{\Gamma}$  (with  $K$  a function of the entropy and the polytropic index  $\Gamma$  remaining constant throughout the collision process), it is conjectured that no prompt collapse can occur for an arbitrary  $\Gamma$  and an arbitrary initial  $K$ . The basic argument is that the potential energy when converted to thermal energy by shock heating is always enough to support the merged object, until neutrino cooling sets in.

The argument based on conservation is appealing, and provides useful understanding for a range of the NS coalescence problems. However there is a major assumption for the argument to go through, namely, the collision

process can be approximated by a quasi-equilibrium process, in two senses: (A) The coalescing matter can be described by one *single* EOS everywhere ( $K$  is a function of time but not space), and (B) whether it collapses or not is determined by hydro-static equilibrium conditions, i.e., whether a stable equilibrium configuration *exists* or not. This quasi-equilibrium assumption is not self-evident for the head-on collision of heavier NSs. It could happen that the coalesced object collapses before it can thermalize, *or* the collision process is so dynamic that even though a stable equilibrium state exists, it is not attained in the collapse process. The final outcome depends on the various time scales in the problem.

**Time Scale Considerations.** We examine this assumption of ‘‘quasi-equilibrium’’ and see if it can be justified under the conditions of (I) and (II) above. We note that the collision process involves many time scales, and there are at least six of them relevant for our present consideration: 1. The time scale associated with the infall velocity:  $t_i = R/V_i$ ,  $R$ =the radius of the NS,  $V_i$ = (infall velocity at the point of contact). 2. The time scale associated with the local sound velocity:  $t_s = R/V_s$ ,  $V_s$ = (sound velocity). 3. The time scale associated with the velocity of the shock (velocity in the rest frame of fluid):  $t_{sh} = R/V_{sh}$ ,  $V_{sh}$ =(shock velocity). 4. The time scale for the merged object to thermalize, in the sense of being describable by one single EOS (same  $K$  everywhere):  $t_e$ . 5. The time scale of neutrino cooling  $t_n$ . 6. The time scale of the gravitational collapse  $t_c$ .

Some comments of these time scales are in order. We focus on the case of two  $1.4 M_{\odot}$  NSs. We model them with a polytropic EOS  $P = K\rho^{\Gamma}$  with a polytropic index of  $\Gamma = 2$ . The initial  $K$  value of the two stars is taken to be  $1.16 \times 10^5 \frac{\text{cm}^5}{\text{g s}^2}$ . (Maximum stable mass of these values of  $K$  and  $\Gamma$  is  $1.46 M_{\odot}$ .) We note that the argument in [2] is applicable to all polytropic models.

For this model,  $V_i$  is (somewhat larger than) the Newtonian value  $\sim 0.28c$ , as can be estimated by  $\sqrt{GM/(2R)}$ ; the diameter of the NSs is about  $2R = 26km$  (the isotropic coordinate radius of this NS is  $9.3km$ , the proper radius is  $R = 13km$ ). Hence the time scale associated with the infall velocity  $t_i$  is about (smaller than)  $0.16 ms$ . To estimate the second time scale  $t_s$ , note that the sound velocity  $V_s$  depends strongly on the dynamical process and the region under consideration. For the model mentioned above, the initial central rest mass density of the NSs is about  $1.5 \times 10^{15} g/cm^3$ ;  $V_s$  there is about  $0.5 c$ . With the density elsewhere initially lower than this value, but higher in some period in the central region of the collision,  $V_s$  varies but is roughly  $0.5c$ . Thus,  $t_s$  is roughly  $0.1ms$ . To estimate the third time scale  $t_{sh}$  requires an estimation of the velocity of the shock  $V_{sh}$  produced in the collision. The locally measured proper velocity of the shock  $V_{sh}$  is higher than, but of the same order of magnitude of, the sound speed  $V_s$  at a fraction of  $c$  in the head-on collision case. Hence  $t_{sh}$  is also of order  $0.1ms$ . These three time scales determine the time scale 4 which is central to our discussion. In near static situation, or when the bulk velocity of matter is small ( $V_i \ll V_s$  and  $V_i \ll V_{sh}$ ),  $t_e$  can be taken to be a few times  $t_s$  or  $t_{sh}$ . On the other hand, its value in a highly dynamic situation with  $V_i$  comparable to  $V_s$  and  $V_{sh}$  is an important issue to be discussed below. The fifth time scale  $t_n$  governs the final settling down of the merged object after  $t_e$ .  $t_n$  is of the order of seconds, orders of magnitude longer than the first four time scales. The gravitational collapse time scale  $t_c$  is in turn controlled by these time scales 1-5. It can be as short as  $t_i$ , or as long as  $t_n$ . For the collision of two  $1.4 M_\odot$  NSs, the merged object would have to collapse after  $t_c$ , if not before, for most of the reasonable EOS. We call collapse that occurs on the first four time scales prompt collapse, and collapse that occurs on a longer time scale, like  $t_n$ , delayed collapse. For more general coalescence processes, there can be other time scales involved, e.g., the time scale of angular momentum transfer  $t_a$ , and the time scale of gravitational wave emission  $t_g$ . However, for the case of head-on collision with the stars falling in from infinity, we expect strong shock heating causing  $t_n$  to be shorter than  $t_g$ . We do not have to consider  $t_a$  and  $t_g$  in our present consideration.

In Shapiro's argument, the time scale 4,  $t_e$ , is implicitly taken to be the shortest time scale in the problem, so that the system can be described by a single EOS at any instant in the collision process. The above discussion suggests that this may not be true for the two  $1.4 M_\odot$  NS collision case. Indeed, the relations between the time scales 1, 2 and 3 strongly affect  $t_e$ . With  $t_i$  comparable to  $t_s$  and  $t_{sh}$ , dynamic effects are important, and  $t_e$  can be longer than  $t_i$ . In particular, with matter falling in at high speed along the axis of the collision, the speed of the shock wave in that direction would be significantly

reduced, until *after*  $t_i$ , delaying "thermalization" of the coalescing objects. For situations like this, arguments based on a uniform EOS throughout the coalescing object cannot be justified. Indeed, when the infalling time scale  $t_i$  is comparable to the other time scales in the process, it could happen that even if a hydrostatic stable equilibrium configuration exist, the dynamics of the system might not lead to that configuration and the time scale of collapse could be as short as  $t_i$ .

Another way of looking at the problem is to imagine we tie the two stars on strings and lower them towards one another in a quasi-stationary fashion while depositing the potential energy extracted back to the two stars. For this case Shapiro's argument would be applicable. However, for a NS collision with the time scales discussed above, one would have to examine the dynamics of the infall to determine whether a prompt or delayed collapse would occur. In short, as both a thermally supported merged object and a black hole can have the same rest mass and total energy, arguments based solely on conservation of mass and energy without taking the dynamics into consideration cannot rule out one outcome from the other.

We note that the above time scale considerations suggests that whether it is a delayed or prompt collapse in head-on collision can depend on the initial NS's configuration. It does *not* imply prompt collapse by itself. To demonstrate that a prompt collapse results, one has to perform a fully relativistic simulation.

Our NCSA/Potsdam/Wash U collaboration is developing a multi-purpose 3D numerical code, "Cactus", for relativistic astrophysics and gravitational wave astronomy. This code contains the Einstein equations coupled to the general relativistic hydrodynamic equations. For a description of various aspects of the code and the NS grand challenge project based on it see [3]. Testbeds and methods for evolving neutron stars have been given in [4], and will not be repeated in this paper. While this multi-purpose code is still under development for various capabilities in treating a broad class of astrophysical scenarios, in this paper we focus on the results obtained by applying this code to the head-on collision problem.

**Simulation results.** We show the  $M = 1.4 M_\odot$  head-on collision case. The stars are modeled as given above. We put the two TOV solutions at a proper distance of  $d = 44 km$  apart (slightly more than  $3 R$  separation) along the z-axis, and boost them towards one another at the speed (as measured at infinity) of  $\sqrt{GM/d}$  (the Newtonian infall velocity). The metric and extrinsic curvature of the two boosted TOV solutions are superimposed by (i) adding the off-diagonal components of the metric, (ii) adding the diagonal components of the metric and subtracting 1, and (iii) adding the components of the extrinsic curvature. The resulting matter distribution, momentum distributions, conformal part of the metric, and transverse traceless part of the extrinsic curvature are then used as input to York's procedure

[5] for determining the initial data, in maximal slicing. With this setup, the initial data satisfies the complete set of Hamiltonian and momentum constraints to high accuracy (terms in the constraints cancel to  $10^{-6}$ ), and physically represent two NSs in head-on collision falling in from infinity, at least up to the Newtonian order. (For initial data setup on the P1N level, see [6]).

The initial data is then evolved with the numerical methods described in [4]. Various singularity avoiding slicings been used and tested against one another (maximal and  $1 + \log$  slicings most extensively), yielding basically the same results. The simulations have been carried out with resolutions ranging from  $\Delta x = 1.48 \text{ km}$  to  $0.246 \text{ km}$  (13 to 76 grid points across each NS, with  $32^3$  to  $192^3$ ) for convergence and accuracy analysis.

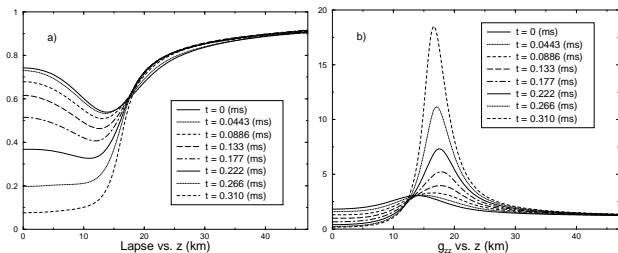


FIG. 1. a) The lapse ( $\alpha$ ) and b)  $g_{zz}$  along the  $z$ -axis are displayed at various times. This simulation used  $192^3$  grid points, with  $\Delta x = 0.246 \text{ km}$ .

In Fig. 1a we show the collapse of the lapse along the  $x = y = 0$  line from  $t = 0 \text{ ms}$  to  $t = 0.31 \text{ ms}$  at intervals of  $0.044 \text{ ms}$ . (With the reflection symmetry across the  $z = 0$  plane and the axisymmetry of the head-on collision, we only need to evolve the first octant.) At  $t = 0.31 \text{ ms}$  the lapse has collapsed significantly.

In Fig. 1b we show the evolution of  $g_{zz}$  along the  $z$  direction ( $x = y = 0$ ). We see the familiar grid stretching effect associated with evolving a black hole.

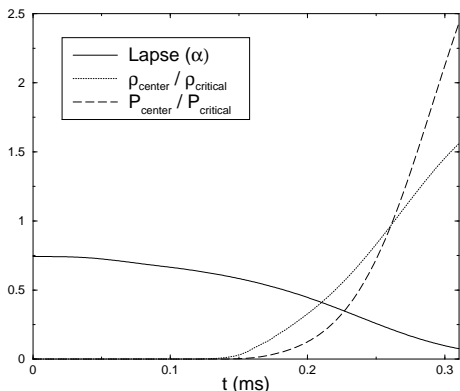


FIG. 2. The evolution of the lapse ( $\alpha$ ), the rest mass density ( $\rho$ ), and the pressure ( $P$ ) in the region centered at the point  $x = y = z = 0$  to the time  $t = 0.31 \text{ ms}$ .

Fig. 2 shows the time development of the lapse, the (proper) rest mass density  $\rho$  and the pressure  $P$  at the ori-

gin, scaled by the critical secular stability values  $\rho_{critical}$  and  $P_{critical}$ , the values beyond which a static TOV solution is unstable to collapse for the given polytropic coefficient  $K$  and index  $\Gamma (= 2)$ . We note that the effective  $K (= P/\rho^2)$  is time dependent due to shock heating. At coordinate time  $t = 0.26 \text{ ms}$  we see that both  $\rho$  and  $P$  surpass  $\rho_{critical}$  and  $P_{critical}$ , indicating a collapse.

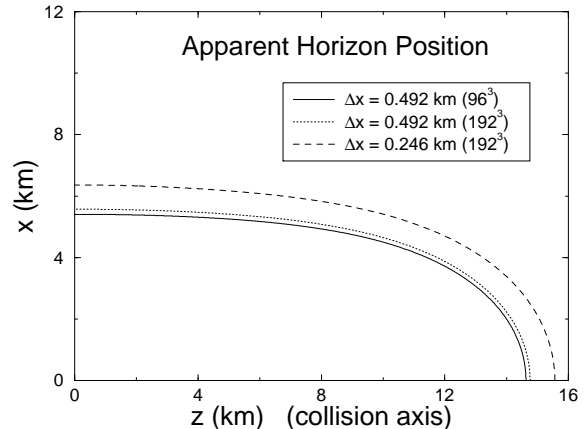


FIG. 3. The position of the AH at different resolutions and outer boundary locations, all at  $t = 0.31 \text{ ms}$ .

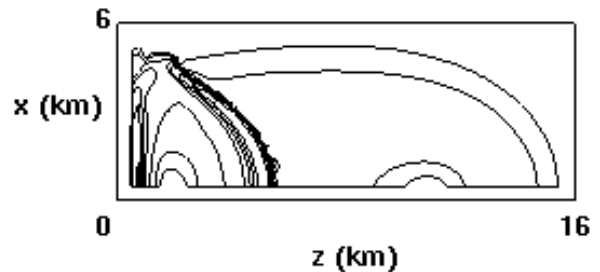


FIG. 4. Contour lines of the log of the gradient of the rest mass density  $\log(\sqrt{\nabla^i(\rho)\nabla_i(\rho)})$ , showing a shock front at coordinate radius  $\sim 4.5 \text{ km}$ .

In Fig.3 we show the position of the apparent horizon (AH). To confirm the location of the AH, convergence tests both in terms of resolution and in terms of location of the computational boundary have been carried out. (For a discussion of the AH finder, see [7]). We have also explicitly determined trapped surfaces bounded by the AH for the positive confirmation of a collapsed region. In Fig. 3, the solid and long dashed lines correspond to the AH locations at resolutions of  $\Delta x = 0.492 \text{ km}$  and  $\Delta x = 0.246 \text{ km}$ , while the dotted line corresponds to  $\Delta x = 0.492 \text{ km}$  but with the outer boundary two times further out. Although the coordinate position of the AH is substantially elongated in the  $z$  direction, the AH is actually quite spherical. The proper circumference on the  $x$ - $y$  plane (equatorial) is close to the circumference on the  $x$ - $z$  plane (polar), with the latter being  $52.9 \pm 1.9 \text{ km}$ .

For comparison,  $4\pi M_{AH}$  is  $52.9 \pm 2.1 \text{ km}$ , where  $M_{AH}$  is the mass of the AH (we note that a substantial part of the matter in the system is enclosed within the AH). Analysis of this in relation to the hoop conjecture will be given elsewhere.

Fig. 4 shows the contour lines in the  $y = 0$  plane of the log of the gradient of the rest mass density  $\log(\sqrt{\nabla^i(\rho)\nabla_i(\rho)})$  at time  $t = 0.31 \text{ ms}$ . We see a sharp peak at a coordinate radius of  $\sim 4.5 \text{ km}$ . The sharp change in rest mass density indicates a shock, stronger in the infalling direction ( $z$ ), while weaker near the equatorial plane. The shock is moderately relativistic with Lorentz factor of about 1.2. The shock is well captured in this 192<sup>3</sup> run with high resolution shock capturing (HRSC) GR-hydro treatment. Comparing to Fig. 3, we see that the shock front is inside the AH in all direction at this time, although it is still moving outward in coordinate location.

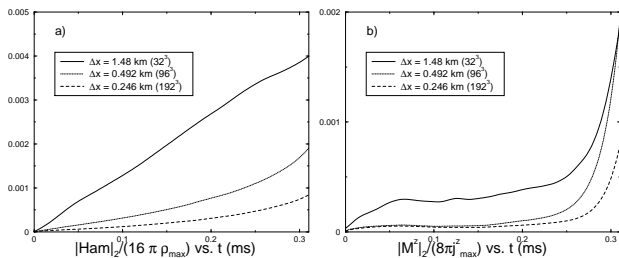


FIG. 5. The evolutions of the  $L2$  norms of (a) the Hamiltonian constraint, and (b) the  $z$ -component of the momentum constraint.

In Fig. 5 we show the convergence of the Hamiltonian and the  $z$ -momentum constraints for a measure of the accuracy of the simulation. The evolution of the  $L2$  norms (integrated squared) of the constraints are scaled by the maximum of the matter terms in the constraints ( $16\pi\rho_{ADM}$  and  $8\pi j_{ADM}^z$  respectively). The solid, dotted, and dashed lines represent the constraints at resolutions  $\Delta x = 1.48 \text{ km}$ ,  $0.492 \text{ km}$  and  $0.246 \text{ km}$  respectively. These *long time scale* convergence tests indicate that our numerical evolution is *stable and convergent* for the time scale of our present problem. Towards the end we see that the error is increasing rapidly; an examination of the spatial distribution of the constraint violations shows that the error is due to the familiar problem of resolving the “grid stretching” peaks of the black hole metric (cf. Fig. 2). An extensive convergence analysis of many of the variables involved in the simulation has been carried out and will be presented in a follow up paper. We have also performed simulations with the initial boost velocity increased by 10% (generating more shock heating) and confirmed that our results are not sensitive to the initial velocity.

With these results we conclude that prompt collapse of the merged object formed in head-on collision infalling

from infinity is possible, under the *same* conditions as in Shapiro’s conjecture.

We have also carried out simulations of head-on collisions of lower mass NSs and have seen cases in which the shocks propagate to cover the whole star and no AH is found, indicating that the collapse would be delayed until radiative cooling. A detailed analysis of the transition point between prompt and delay collapse is computationally expensive with our 3D code used to carry out the present analysis. A 2D version of the present treatment is being developed with this specific application in mind.

**Conclusions.** We pointed out that there is an assumption in Shapiro’s conjecture, namely, the head-on collision process is in quasi-equilibrium (in the sense of (A) and (B) above). We showed that this may not be true for the collision of two  $1.4 M_\odot$  NS’s. We substantiated our argument with a simulation solving the full set of the coupled Einstein and general relativistic hydrodynamic equations. We confirmed the prompt formation of a black hole in the infalling time scale  $t_i$  with an apparent horizon found  $0.16 \text{ ms}$  after the point of contact.

In this paper we concentrate on the head-on collision process under the same conditions as in Shapiro’s conjecture. As the time scale argument given above is rather general, and in particular does not depend on the polytropic EOS, we expect the same argument to be applicable to more general situations. An investigation of the prompt vs. delayed collapse problem of head-on collisions with realistic EOSs, more realistic initial conditions (initial data setup with Post-Newtonian formulation), and with a determination of the critical point between delayed vs. prompt collapse will be given in follow up papers.

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